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The effect of a depth discontinuity on Kelvin wave diffraction

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The linearized problem of Kelvin wave diffraction at the sharp edge of a thin semi-infinite barrier on a rotating earth is considered in the case when a discontinuity in depth stems from the edge in a direction parallel to the barrier. The solution in closed form is obtained using the Wiener–Hopf technique.

It is shown that the direct effect of the depth discontinuity is always to divert additional energy away from the barrier, in the form of outgoing cylindrical waves if the period is small, and provided conditions are suitable as double Kelvin waves if the period is sufficiently long. Results suggest that Kelvin waves which are the most persistent in following an irregular coastline bounding a sea of abruptly varying depth are of intermediate period.

1. Introduction

Recent papers by Buchwald (1968) and Packham & Williams (1968) have dealt with the diffraction of Kelvin waves at a sharp bend bounding a sea of uniform depth. If $\tau > 1$, where τ is the wave period in pendulum-days, the incident Kelvin waves are diffracted with no loss in energy. However if $\tau < 1$, it is possible for energy to be transmitted to infinity by means of outgoing cylindrical waves, which leads in consequence to a diminution in amplitude of the diffracted Kelvin wave. This effect becomes more marked the smaller the value of τ taken.

One purpose of the present paper is to study, in the case $\tau < 1$, a typical diffraction problem, with the inclusion of an abrupt change in depth in order to evaluate the effect of the depth discontinuity on diverting energy away from the boundaries. In addition, the corresponding case of waves of long period $\tau > 1$ is of some interest. Longuet-Higgins (1968) shows that a system of double Kelvin waves of long period is capable of travelling along the depth discontinuity provided the circumstances are favourable and the shallower water is to the right of the direction of propagation in the northern hemisphere and to the left in the southern hemisphere. Waves of this type would be trapped by the discontinuity if a local disturbance such as a storm contained components of long period. This question has been considered by Mysak (1969) for the case of either a transient or a time-periodic wind stress which is suddenly applied to an otherwise undisturbed sea surface. In the present paper the generation of double Kelvin waves is also considered, but here the long period energy is to be supplied by a Kelvin wave moving along a coastline in the vicinity of the abrupt change in depth. This is a situation which can be envisaged as taking place at many places on the earth, as for example a Kelvin wave moving southwards down the Pacific coast of South America emerging into shallower water as it rounds Cape Horn and the Falkland Islands. Again a northward moving Kelvin wave along the Californian coast could in theory set up a westward moving double Kelvin wave along the Mendocino escarpment since the shallower water lies to the north.

2. Formulation of the problem

Consider a horizontal region of water rotating about a vertical axis with angular velocity $\frac{1}{2}f$ in the same sense as in the northern hemisphere. Horizontal rectangular co-ordinates (x, y) are now defined and it is assumed that the x axis

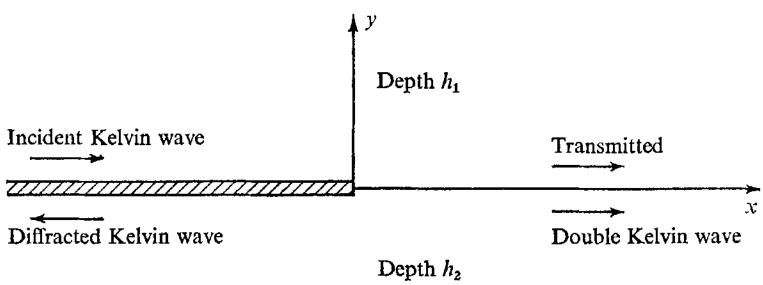

FIGURE 1. Physical plane when a double Kelvin wave is produced.

divides the flow into two regions of constant depth h_n, adopting the convention that $n = 1$ for $y > 0$, and $n = 2$ for $y < 0$. Separating the regions along the x axis, there is a thin solid barrier present for $x < 0$, and a discontinuity in depth for $x > 0$, as illustrated in figure 1.

Assuming that the motion is described by the linearized shallow water theory the equations of motion are

$$\frac{\partial u_n}{\partial t} - f v_n = -g \frac{\partial \zeta_n}{\partial x}, \tag{2.1}$$

$$\frac{\partial v_n}{\partial t} + f u_n = -g \frac{\partial \zeta_n}{\partial y}, \tag{2.2}$$

and the equation of continuity is

$$\frac{\partial \zeta_n}{\partial t} = -h_n \left(\frac{\partial u_n}{\partial x} + \frac{\partial v_n}{\partial y} \right). \tag{2.3}$$

Here (u_n, v_n) are the velocity components at time t in the (x, y) directions, ζ_n is the free surface elevation above its mean level, and g is the acceleration of gravity. Allowing for a time dependence of $e^{-i\sigma t}$ then (2.1), (2.2) yield,

$$u_n = \frac{g}{f^2 - \sigma^2} \left\{ -f \frac{\partial \zeta_n}{\partial y} + i\sigma \frac{\partial \zeta_n}{\partial x} \right\}, \tag{2.4}$$

$$v_n = \frac{g}{f^2 - \sigma^2} \left\{ i\sigma \frac{\partial \zeta_n}{\partial y} + f \frac{\partial \zeta_n}{\partial x} \right\}. \tag{2.5}$$

Inserting u_n, v_n in (2.3) gives

$$\left.\begin{aligned} &\frac{\partial^2\zeta_n}{\partial x^2}+\frac{\partial^2\zeta_n}{\partial y^2}-k_n^2\,\zeta_n = 0, \\ \text{defining} \qquad &k_n^2 = \frac{(f^2-\sigma^2)}{c_n^2}, \quad \text{where} \quad c_n = (gh_n)^{\frac{1}{2}}. \end{aligned}\right\} \tag{2.6}$$

At the solid barrier, the normal fluid velocity is zero, which gives rise to the boundary condition for $n = 1, 2$

$$v_n(x, 0) = 0, \quad \text{when} \quad x < 0. \tag{2.7}$$

At the discontinuity, following Longuet-Higgins (1968), the surface elevation and the normal component of the flux are taken to be continuous, and thus

$$[\zeta_n]_1^2 = 0, \quad [h_n v_n]_1^2 = 0 \quad \text{for} \quad x > 0. \qquad (2.8), (2.9)$$

It is now assumed that

$$\zeta_n = \exp\{(i\sigma x - fy)/c_1\} + \phi_1 \quad \text{for} \quad n = 1, y > 0, \tag{2.10a}$$

$$= \phi_2 \quad \text{for} \quad n = 2, y < 0. \tag{2.10b}$$

The first term on the right-hand side of (2.10a) represents an incident Kelvin wave for $y > 0$ travelling from $x = -\infty$ left to right along the solid barrier with speed c_1 (as shown in figure 1). The resulting motion is then conjectured to be: (i) a diffracted Kelvin wave for $y < 0$ moving right to left along the solid barrier to $x = -\infty$ with speed c_2 giving a contribution to ζ_2 of the form

$$\phi_2 = A\exp\{(-i\sigma x - fy)/c_2\}, \tag{2.11}$$

and (ii) a transmitted double Kelvin wave travelling along the depth discontinuity to $x = +\infty$ of the form

$$\phi_n = B\exp\{imx \mp (k_n^2 + m^2)^{\frac{1}{2}}\,y\} \tag{2.12}$$

provided m is a real root of the equation

$$(h_1 - h_2)\,mf = h_1\sigma(m^2+k_1^2)^{\frac{1}{2}} + h_2\sigma(m^2+k_2^2)^{\frac{1}{2}}. \tag{2.13}$$

In (2.12) the convention is introduced that the upper sign is taken for $n = 1$, and the lower sign for $n = 2$. The relations (2.12) and (2.13) both stem directly from the work of Longuet-Higgins (1968) on double Kelvin waves. In the same work it is shown that these waves only arise if the depth ratio $\gamma = h_1/h_2 > 1$, and provided $f > \sigma(\gamma+1)/(\gamma-1)$. In general, however, in this paper f is not restricted, but may take any value.

Thus the problem is reduced to that of finding the solution ζ_n of (2.6) satisfying the boundary conditions (2.7), (2.8) and (2.9) which will lead to a determination of $|A|$, $|B|$, the amplitudes of the waves directly associated with the barrier and discontinuity respectively at infinity. To satisfy the radiation condition at infinity it will be sufficient to assume that σ has a positive imaginary part σ_i which will eventually be allowed to go to zero to obtain the desired solution which is periodic in time. The incident wave with $\sigma_i > 0$ may then be thought of growing from small values like $e^{\sigma_i t}$, while at the same time having an amplitude decreasing as $e^{-\sigma_i x}$ in the positive x direction. The secondary waves due to the presence of the barrier and the depth discontinuity will then die out at large distances from $x = 0$. This may be seen from equations (2.10) to (2.12) since ϕ_2 is $O(e^{\sigma_i x/c_2})$ as $x \to -\infty$ and ϕ_n is $O(e^{-m_i x})$ or $O(e^{-\sigma_i x/c_1})$ as $x \to +\infty$, where m_i is the imaginary part of m. Here it has been anticipated that ϕ_1 must incorporate a wave-like component which cancels out the incident Kelvin wave as $x \to +\infty$. In addition, $m_i > 0$, for small $\sigma_i > 0$, since $d\sigma/dm \to c_G$ as $\sigma_i \to +0$, where c_G is positive being the group velocity of the double Kelvin waves.

3. Use of Fourier transforms

Following a notation close to that adopted by Noble (1958) left-handed and right-handed Fourier transforms are defined as

$$\Phi_n(\alpha, y) = \Phi_n^+(\alpha, y) + \Phi_n^-(\alpha, y), \tag{3.1}$$

where
$$\left.\begin{aligned} \Phi_n^+(\alpha, y) &= \frac{1}{(2\pi)^{\frac{1}{2}}}\int_0^\infty \phi_n(x, y)\, e^{i\alpha x}\, dx, \\ \Phi_n^-(\alpha, y) &= \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^0 \phi_n(x, y)\, e^{i\alpha x}\, dx, \end{aligned}\right\} \tag{3.2}$$
and

for $\alpha = \alpha_r + i\alpha_i$.

Transforming (2.6) then

$$d^2\Phi_n/dy^2 - \tau_n^2\, \Phi_n = 0, \tag{3.3}$$

where $\tau_n = (\alpha^2 + k_n^2)^{\frac{1}{2}}$.

The solution of (3.3) satisfying $\Phi_n \to 0$, as $y \to \pm\infty$ is

$$\Phi_n(\alpha, y) = D_n(\alpha)\, e^{\mp\tau_n y}. \tag{3.4}$$

Thus $\Phi_n(\alpha, y)$ has branch points at $\alpha = \pm ik_1, \pm ik_2$. Defining δ to be the minimum value of m_i, σ_i/c_n, Re (k_n), then it follows that $\Phi_n^+(\alpha, y)$ is a regular function of α for all $\alpha_i > -\delta$ while $\Phi_n^-(\alpha, y)$ is a regular function for all $\alpha_i < \delta$.

Considering now the boundary conditions, then on transforming using (2.10), (2.8) becomes

$$\Phi_1^+(\alpha, 0) + \frac{i}{(2\pi)^{\frac{1}{2}}}\frac{1}{(\alpha + (\sigma/c_1))} = \Phi_2^+(\alpha, 0). \tag{3.5}$$

By means of (2.5), (2.10) and (3.2), the boundary condition (2.7) may be transformed to the form

$$i\sigma\{d\Phi_n^-/dy\}_{y=0} - i\alpha f \Phi_n^-(\alpha, 0) + f\phi_n(-0, 0) = 0. \tag{3.6}$$

Treating the condition (2.9) in a similar manner it can be written in a form which defines a function $\psi^+(\alpha)$ regular for $\alpha_i > -\delta$, i.e.

$$\psi^+(\alpha) = h_n[i\sigma\{d\Phi_n^+/dy\}_{y=0} - i\alpha f\Phi_n^+(\alpha, 0) - f\phi_n(+0, 0)] \tag{3.7}$$

for $n = 1$ or 2.

Assuming the continuity condition $\phi_n(-0, 0) = \phi_n(+0, 0)$ (3.6) and (3.7) may now be combined to give

$$\psi^+(\alpha) = h_n[i\sigma\{d\Phi_n/dy\}_{y=0} - i\alpha f\Phi_n(\alpha, 0)] \tag{3.8}$$

for $n = 1$ or 2.

The equation (3.8) is now in a form where (3.4) can be inserted directly and yields

$$\psi^{+}(\alpha) = h_n D_n[\mp i\sigma\tau_n - i\alpha f] \tag{3.9}$$

for $n = 1$ or 2.

Now (3.1) and (3.4) give on $y = 0$,

$$D_n(\alpha) = \Phi_n^{+}(\alpha, 0) + \Phi_n^{-}(\alpha, 0). \tag{3.10}$$

Thus it follows that

$$D_2(\alpha) - D_1(\alpha) = \Phi_2^{+}(\alpha, 0) - \Phi_1^{+}(\alpha, 0) + \Phi_2^{-}(\alpha, 0) - \Phi_1^{-}(\alpha, 0).$$

Using (3.5) then

$$D_2(\alpha) - D_1(\alpha) = \frac{i}{(2\pi)^{\frac{1}{2}}} \frac{1}{(\alpha + (\sigma/c_1))} + \Gamma^{-}(\alpha), \tag{3.11}$$

where

$$\Gamma^{-}(\alpha) = \Phi_2^{-}(\alpha, 0) - \Phi_1^{-}(\alpha, 0) \tag{3.12}$$

is a regular function of α for $\alpha_i < \delta$.

Finally eliminating $D_1(\alpha)$ and $D_2(\alpha)$ in (3.11) by means of (3.9) it may be deduced that

$$\psi^{+}(\alpha)\, I(\alpha) = -\frac{i}{(2\pi)^{\frac{1}{2}}} \frac{1}{(\alpha + (\sigma/c_1))} + i\Gamma^{-}(\alpha), \tag{3.13}$$

where

$$I(\alpha) = \frac{\alpha f - \sigma\tau_2 - \gamma(\alpha f + \sigma\tau_1)}{\gamma(\alpha f - \sigma\tau_2)(\alpha f + \sigma\tau_1)}. \tag{3.14}$$

The equation (3.13) contains two unknown functions $\psi^{+}(\alpha)$ and $\Gamma^{-}(\alpha)$. Nevertheless, since each function is regular in its respective half plane, the equation is of a form in which each function may be determined by the Wiener–Hopf method, provided $I(\alpha)$ can be successfully factorized into two parts separately regular in the different half planes.

4. The factorization of $I(\alpha)$

It can be seen from (3.14) that $I(\alpha)$ vanishes when $\alpha = -m$ if m satisfies (2.13), the condition for a double Kelvin wave. In addition, the factors in the denominator of $I(\alpha)$ make the denominator vanish when $\alpha f \pm \sigma\tau_n = 0$, i.e. if $\alpha = \mp\sigma/c_n$. It is convenient therefore to extract these factors and to define a function $K(\alpha)$, regular for $-\delta < \alpha < \delta$, by means of the relation

$$I(\alpha) = \frac{(\alpha + m)\, HK(\alpha)}{(\alpha - (\sigma/c_2))(\alpha + (\sigma/c_1))}, \tag{4.1}$$

where

$$H = \frac{(f-\sigma) - \gamma(f+\sigma)}{\gamma(f^2 - \sigma^2)} \tag{4.2}$$

is a constant.

By its definition $K(\alpha)$ cannot vanish or become infinite on the real α axis as $\delta \to 0$. Moreover, as $\alpha \to +\infty$, $K(\alpha) \to 1$, while if $\alpha \to -\infty$, $K(\alpha) \to J$, where

$$J = \frac{(f+\sigma) - (f-\sigma)\gamma}{(f-\sigma) - (f+\sigma)\gamma}. \tag{4.3}$$

In (4.1) m is chosen to be the real root of (2.13) when it exists, but if it does not exist, i.e. if $f < \sigma(\gamma+1)/(\gamma-1)$ for $\gamma > 1$ or if $\gamma < 1$ then m is chosen to be any complex constant such that $m_i > \delta$. It will be seen later for this case that the arbitrary choice of m, whose introduction facilitates the factorization of $I(\alpha)$ will cancel out in obtaining results of physical interest.

To factorize $K(\alpha)$ it is necessary to adapt slightly theorem C given by Noble (1958) to allow for the fact that as $\alpha \to -\infty$, $K(\alpha) \to J \neq 1$, as well as to ensure convergence of all integrals appearing in the analysis. To this end the function $\log K(\beta)/(\beta-\alpha)$ is integrated in the β plane over a rectangle, containing the point $\beta = \alpha$, with corners at $\beta = \pm i\delta_0 - R$ and $\pm i\delta_0 + R_1$, where $\delta_0 < \delta$. In addition, the function $\log J/(\beta-\alpha)$ is integrated over a rectangle with corners at

$$\beta = \pm i\delta_0 - R \quad \text{and} \quad \beta = \pm i\delta_0.$$

Cauchy's integral theorem is then applied and allowing R and $R_1 \to \infty$ it follows that

$$K(\alpha) = K^+(\alpha)\,K^-(\alpha), \tag{4.4}$$

where choosing the appropriate upper or lower signs,

$$K^\pm(\alpha) = \exp\left\{\pm\frac{1}{2\pi i}\left[\int_{\mp i\delta_0-\infty}^{\mp i\delta_0} \log\left(\frac{K}{J}\right)\frac{d\beta}{\beta-\alpha} + \int_{\mp i\delta_0}^{\mp i\delta_0+\infty} \log K\,\frac{d\beta}{\beta-\alpha} - \log J \log(\alpha \pm i\delta_0)\right]\right\}. \tag{4.5}$$

Here the convergence of the integrals has been ensured and $K^+(\alpha)$ is regular for $\alpha_i > -\delta_0 > -\delta$ and $K^-(\alpha)$ is regular for $\alpha_i < \delta_0 < \delta$.

5. Solution by the Wiener–Hopf technique

Using (4.1) and (4.4), (3.13) may now be rewritten in the form

$$\psi^+(\alpha)\,H\,\frac{\alpha+m}{\alpha+(\sigma/c_1)}\,K^+(\alpha) - \frac{\sigma\left(\dfrac{1}{c_1}+\dfrac{1}{c_2}\right)}{(2\pi)^{\frac{1}{2}}\,K^-\left(-\dfrac{\sigma}{c_1}\right)\left(\alpha+\dfrac{\sigma}{c_1}\right)}$$

$$= \frac{i\Gamma^-(\alpha)}{K^-(\alpha)}\left(\alpha-\frac{\sigma}{c_2}\right) - \frac{1}{(2\pi)^{\frac{1}{2}}}\,\frac{1}{\left(\alpha+\dfrac{\sigma}{c_1}\right)}\left\{\frac{\alpha-\dfrac{\sigma}{c_2}}{K^-(\alpha)} + \frac{\sigma\left(\dfrac{1}{c_1}+\dfrac{1}{c_2}\right)}{K^-\left(-\dfrac{\sigma}{c_1}\right)}\right\}, \tag{5.1}$$

in which the left-hand side is regular for $\alpha_i > -\delta_0$, and the right-hand side is regular for $\alpha_i < \delta_0$. Each term on the left-hand side of (5.1) tends to zero as $\alpha_i \to \infty$, and by the usual arguments based on Louville's theorem it may be deduced that each side of (5.1) is identically zero.

Thus it follows that

$$\psi^+(\alpha) = \frac{c_1+c_2}{(2\pi)^{\frac{1}{2}}\,HK^-(-\sigma/c_1)\,K^+(\alpha)\,(\alpha+m)}. \tag{5.2}$$

Finally substituting for $\psi^+(\alpha)$ in (3.9) to find $D_n(\alpha)$, and then taking the Fourier inverse of (3.4) yields the required solution,

$$\phi_n = \frac{c_1+c_2}{2\pi H h_n}\int_{-\infty}^{\infty}\frac{\exp\{\mp\tau_n y - i\alpha x\}\,d\alpha}{K^-(-\sigma/c_1)\,K^+(\alpha)\,(\alpha+m)\,(\mp i\sigma\tau_n - i\alpha f)}. \tag{5.3}$$

6. The diffracted and transmitted waves

Waves with non-zero amplitudes at infinity will arise only from those poles of the integrand in (5.3) which approach the real axis as $\sigma_i \to 0$. Recalling the radiation condition the residues at these poles must be evaluated before the limit $\sigma_i \to 0$ is applied. Before considering the contributions from the obvious poles $\alpha = \mp\sigma/c_n$, and $\alpha = -m$ (if m is real) it is necessary to inquire if the function $K^+(\alpha)$ which occurs in the integrand of (5.3) has any zeros on the real α axis as $\sigma_i \to 0$, leading to further wave-like contributions.

From (4.5), provided α is real and allowing σ_i, and thus δ_0, to tend to zero, then

$$K^-(-\sigma/c_1)\,K^+(\alpha) = \exp\left\{\frac{1}{2\pi i}\int_\Gamma \log K(\beta)\left(\frac{1}{\beta-\alpha} - \frac{1}{\beta+(\sigma/c_1)}\right)d\beta\right\}. \tag{6.1}$$

If $f > \sigma$ ($\sigma_i = 0$) the path of integration Γ is along the real β axis except where it passes below $\beta = \alpha$ and above $\beta = -\sigma/c_1$ along small semi-circular paths. If $f < \sigma$, as $\sigma_i \to 0$ the branch points of $K(\beta)$ approach the real axis at $\beta = \pm\kappa_1, \pm\kappa_2$, for $n = 1, 2$, where $\kappa_n^2 = -k_n^2 = (\sigma^2 - f^2)/c_n^2$. In taking Γ these branch points must be excluded by passing below at $\beta = -k_1, -k_2$ and above at $\beta = k_1, k_2$.

Considering now the residue from the pole $\beta = \alpha$ in the integrand of (6.1) it may be directly inferred that as α varies along the real axis $K^+(\alpha)$ can only vanish where the function $K(\alpha)$ has a zero. Since $K(\alpha)$ has been so constructed to have no such zeros on the real axis, it is concluded that no waves of non-zero amplitude for large x may originate from the $K^+(\alpha)$ term. This conclusion is further illustrated by results (6.3), (6.5) obtained below.

Returning to (5.3) the waves occurring for large x are now considered in detail. The pole at $\alpha = -\sigma/c_1$ contributes as $x \to \infty$ for $y > 0$. As to be expected its residue yields a Kelvin wave in this region which exactly cancels out the incident Kelvin wave given in (2.10*a*).

The diffracted Kelvin wave which takes the form (2.11) may be obtained from the pole $\alpha = \sigma/c_2$ yielding

$$A = -\frac{\gamma(c_1+c_2)}{h_2 H(\gamma^2-\sigma^2)\,(m+c_2)\,K^-(-\sigma/c_1)\,K^+(\sigma/c_2)}. \tag{6.2}$$

In (6.2) it is now permissible to allow $\sigma_i \to 0$ and take σ to be entirely real. The amplitude $|A|$ of the diffracted wave is of particular physical interest and three cases arise in its determination.

(i) If $f > \sigma(\gamma+1)/(\gamma-1)$ for $\gamma > 1$, as $\sigma_i \to 0$, m is real and $K(\beta)$ is real, bounded, and has no zeros for all real β. Thus by (6.1)

$$|K^-(-\sigma/c_1)\,K^+(\alpha)| = (K(-\sigma/c_1)\,K(\alpha))^{\frac{1}{2}}, \tag{6.3}$$

if α is real, since the only contributions to this modulus emanate from the semi-circular arcs of Γ excluding the poles $\beta = \alpha$, $\beta = -\sigma/c_1$.

(ii) If $\sigma < f < \sigma(\gamma+1)/(\gamma-1)$ for $\gamma < 1$, or if $\sigma < f$ for $\gamma > 1$, then m has been chosen such that its imaginary value $m_i > 0$. Now introduce a function

$$L(\alpha) = (\alpha+m)\,K(\alpha)$$

which is real for all real α. Then since for real α

$$\exp\left\{\frac{1}{2\pi i}\int_\Gamma \log(\beta+m)\left\{\frac{1}{\beta-\alpha}-\frac{1}{\beta+(\sigma/c_1)}\right\}d\beta\right\} = \alpha+m,$$

it follows from (6.1) that

$$K^-\left(-\frac{\sigma}{c_1}\right)K^+(\alpha)\,(\alpha+m) = \exp\left\{\frac{1}{2\pi i}\int_\Gamma \log L(\beta)\left\{\frac{1}{\beta-\alpha}-\frac{1}{\beta+(\sigma/c_1)}\right\}d\beta\right\}. \quad (6.4)$$

With similar reasoning to case (i), therefore for α real it follows that

$$|K^-(-\sigma/c_1)\,K^+(\alpha)\,(\alpha+m)| = (L(-\sigma/c_1)\,L(\alpha))^{\frac{1}{2}}. \quad (6.5)$$

(iii) If $f < \sigma$, for all γ, $m_i > 0$ and (6.4) again follows but now $L(\beta)$ is no longer real on the real β axis due to the branch points at $\beta = \pm\kappa_1, \pm\kappa_2$. The result corresponding to (6.5) is therefore more complicated. Only the case $\alpha > 0$ was in fact required and this was more conveniently obtained by contour integration in terms of an integral over the imaginary β axis leading to the result (6.6c) given below.

To relate the present work directly to that of Longuet-Higgins the following non-dimensional variables are defined, $\tau = f/\sigma$, $\epsilon = f^2/c_1^2 m^2$, where τ is the wave period measured in pendulum-days and ϵ is a parameter dependent on m the wave-number of the double Kelvin wave. Putting $\alpha = \sigma/c_2$ in cases (i)–(iii), (6.2) yields for the diffracted wave amplitude

$$|A| = \left\{\frac{\gamma(\tau-\epsilon^{\frac{1}{2}})}{\tau+(\epsilon\gamma)^{\frac{1}{2}}}\right\}^{\frac{1}{2}} \quad \text{for} \quad \text{(i)}, \quad (6.6a)$$

$$= \gamma^{\frac{1}{2}} \quad \text{for} \quad \text{(ii)}, \quad (6.6b)$$

$$= 2\tau\gamma^{\frac{3}{4}}\exp(M(\gamma,\tau)) \quad \text{for} \quad \text{(iii)}. \quad (6.6c)$$

In (6.6c) $$M(\gamma,\tau) = \frac{-(1+\gamma^{\frac{1}{2}})}{2\pi}\int_0^\infty \frac{s^2+\gamma^{\frac{1}{2}}}{(s^2+1)(s^2+\gamma)}\log N(s,\gamma,\tau)\,ds,$$

where $$N(s) = \gamma(s^2+1-\tau^2)^{\frac{1}{2}}+(s^2+(1-\tau^2)\gamma)^{\frac{1}{2}}-s^2(\gamma-1)^2\tau^2.$$

When the depth discontinuity vanishes, i.e. when $\gamma = 1$, the integral in (6.6c) may be evaluated exactly yielding

$$|A| = \left\{\frac{(1-\tau^2)^{\frac{1}{2}}-1}{(1-\tau^2)^{\frac{1}{2}}+1}\right\}^{\frac{1}{2}}.$$

This result agrees with the work of Packham & Williams when $\tau < 1$ in the case of a wedge of angle 2π.

Finally attention is turned to the double Kelvin wave, which occurs only in case (i). Evaluating the residue at $\alpha = -m$, the wave is of the form (2.12) where

$$B = \frac{c_1+c_2}{Hh_n(\gamma m+\sigma(k_n^2+m^2)^{\frac{1}{2}})\,K^-(-\sigma/c_1)\,K^+(-m)}, \quad (6.7)$$

where n may take either value $n = 1$ or 2 in virtue of the dispersion relation (2.13). Taking the limit $\sigma_i \to 0$ and applying (6.3) for $\alpha = -m$ then

$$|B| = \frac{c_1 + c_2}{Hh_n(\gamma m + \sigma(K_n^2 + m^2)^{\frac{1}{2}})\,(K(-\sigma/c_1)\,K(-m))^{\frac{1}{2}}}.$$

Thus in terms of non-dimensional variables

$$|B| = \left\{\frac{(\tau^2 - 1)\,\epsilon^{\frac{1}{2}}\,(1 + \gamma^{\frac{1}{2}})}{\tau^2(\tau + (\epsilon\gamma)^{\frac{1}{2}})\left(1 - \dfrac{1}{\gamma} - \dfrac{1}{\gamma T} - \dfrac{1}{S}\right)}\right\}^{\frac{1}{2}}, \tag{6.8}$$

where $S = [\epsilon(\tau^2 - 1) + \tau^2]^{\frac{1}{2}}$ and $T = [\gamma\epsilon(\tau^2 - 1) + \tau^2]^{\frac{1}{2}}$. Using (2.13) the variables τ, γ, ϵ are related by the dispersion relation

$$\gamma S + T - (\gamma - 1)\,\tau^2 = 0, \tag{6.9}$$

which is illustrated in figure 3 of Longuet-Higgins (1968). As $\epsilon \to 0$, it can be deduced from (6.9) that $\tau \to (\gamma + 1)/(\gamma - 1)$, while from (6.8)

$$|B| \sim \left(\frac{1 + \gamma^{\frac{1}{2}}}{2\epsilon^{\frac{1}{2}}}\right)^{\frac{1}{2}}. \tag{6.10}$$

This completes details of those waves which have finite amplitude at infinity. However, as mentioned in § 1 if $\tau < 1$, a system of cylindrical waves can be set up which propagate energy away from the boundary. These waves may be estimated at large distances from the boundary by Kelvin's method of stationary phase. The contributions arise from (5.3) on integrating between the branch points $\alpha = \pm\kappa_n$ for $n = 1, 2$, putting $\tau_n = -i(\kappa_n^2 - \alpha^2)^{\frac{1}{2}}$ and are given by

$$\zeta_n = \frac{(c_1 + c_2)\sin\theta\exp\{\pm i(\kappa_n r - \frac{1}{4}\pi)\}}{Hh_n(\sigma\sin\theta - if\cos\theta)\,K^-(-\sigma/c_1)\,K^+(\pm\kappa_n\cos\theta)\,(m \mp \kappa_n\cos\theta)}\frac{1}{(\kappa_n r)^{\frac{1}{2}}} + O\left(\frac{1}{r}\right),$$

in terms of polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$. The amplitude of these waves then follows as before using (6.4).

7. Energy flux

For $\tau > 1$, in cases (i) and (ii), the flux of energy to infinity must be entirely accounted for by the Kelvin waves and by the double Kelvin wave when present. Invoking the conservation of this energy provided a useful check on both the accuracy of the calculations carried out, described in § 8, as well as on the validity of the theory. This was especially pertinent in case (i). The situation is different in case (iii), when $\tau < 1$, since a proportion of the energy is carried by outgoing cylindrical waves of vanishing amplitude at infinity, which are not dependent on the boundaries of the flow for their maintenance.

Considering the double Kelvin wave, the sum of the kinetic and potential energy per unit length in the x direction transmitted as $x \to \infty$ is

$$E_T = \frac{m\rho}{4\pi}\int_{-\infty}^{\infty} dy \int_0^{2\pi/m} [(u_n^2 + v_n^2)\,h_n + g\zeta_n^2]\,dx, \tag{7.1}$$

putting $n = 1$ for $y > 0$, and $n = 2$ for $y < 0$. Using (2.4), (2.5), (2.10) and (2.12), then

$$E_T = \frac{E\,|B|^2}{2\epsilon^{\frac{1}{2}}(\tau^2-1)^2\,\gamma ST}$$
$$\times\{\gamma\epsilon(\tau^2-1)^2(T+S)+(\tau^2+1)(\tau^2+T^2)(\gamma T-S)-4\tau^2 ST(\gamma-1)\} \quad (7.2)$$

writing $\frac{1}{2}\rho g c_1^2/\sigma^2 = E$. This energy travels with the group velocity of the double Kelvin waves

$$c_G = \frac{d\sigma}{dm} = \frac{c_1\epsilon((\gamma-1)\tau TS - f(S+\gamma T))}{(\gamma-1)\tau^2 TS-\epsilon^{\frac{3}{2}}(T+\gamma S)}. \quad (7.3)$$

Similar determinations of the total energy per unit length may be made for each Kelvin wave giving

$$E_I = \frac{1}{\tau}E, \quad E_D = \frac{|A|^2 E}{\tau\gamma^{\frac{1}{2}}}$$

for the incident and diffracted waves respectively. Unlike the double Kelvin wave, the energy is always shared equally between the potential and kinetic energy in a Kelvin wave. The Kelvin waves are not dispersive and thus the energies E_I and E_D travel with velocities c_1 and c_2 respectively. The conservation of energy flux for the wave system in region I may therefore be expressed as

$$E_I c_1 = E_D c_2 + E_T c_G. \quad (7.4)$$

8. Conclusions

The flux ratio of diffracted energy to incident energy $F = E_D c_2/E_I c_1$ is given in figure 2 for values of γ in the cases $\tau = f/\sigma = 0{\cdot}6, 0{\cdot}8, 0{\cdot}97, 1{\cdot}3, 1{\cdot}5, 2, 5$. In case (ii) if $\gamma < 1$, $\tau > 1$, or if $\gamma > 1$, $(\gamma+1)/(\gamma-1) > \tau > 1$, then $F = 1$ and all the energy is diffracted, since no Kelvin wave, or any other wave allowing transmission of energy to infinity, is possible. The amplitude $|A|$ of the diffracted Kelvin wave is thus the greatest possible, i.e. $|A| = \gamma^{\frac{1}{2}}$. In case (iii) if $\tau < 1$ for all γ, figure 2 shows that as τ decreases so does F, the energy associated with the diffracted wave. The balance of the energy must be transmitted away from the boundary in the form of cylindrical waves. This result is in accordance with the work of Buchwald and Packham & Williams on Kelvin wave diffraction at sharp bends. In the present work there is an additional factor to take into account, i.e. the occurrence of a discontinuity in depth. Figure 2 shows that if a depth discontinuity exists ($\gamma \neq 1$) it causes a further reduction in diffracted energy, increasing the transmission of energy away from the boundary.

If $\tau < 0{\cdot}8$, the energy dissipated from the boundary is already considerable amounting to about 75% of the incident energy even with no discontinuity present. The bend presented to the incident wave here is of course the sharpest possible. Nevertheless, it does suggest that if a storm takes place in the vicinity of an irregular coastline producing Kelvin wave disturbances of many frequencies, those waves with large frequency will tend to be eliminated more readily from following the coastline, the tendency being enhanced by any abrupt changes in depth present. This result is of course irrespective of any other effects, e.g. due to viscosity or wave breaking.

In case (i) if $\gamma > 1$, $\tau > (\gamma+1)/(\gamma-1)$ no cylindrical waves are possible, but now the discontinuity provides an additional outlet for energy transmission in the form of double Kelvin waves. From figure 2 it may be seen that for given discontinuity as τ is reduced more energy flux is channelled into the diffracted Kelvin wave, a variation which is the reverse of that found in case (iii). Like the previous case this conclusion would be expected to have a general validity independently of the geometry of the boundaries and discontinuities present. For a coastline bounding a sea with many changes in depth, the Kelvin waves

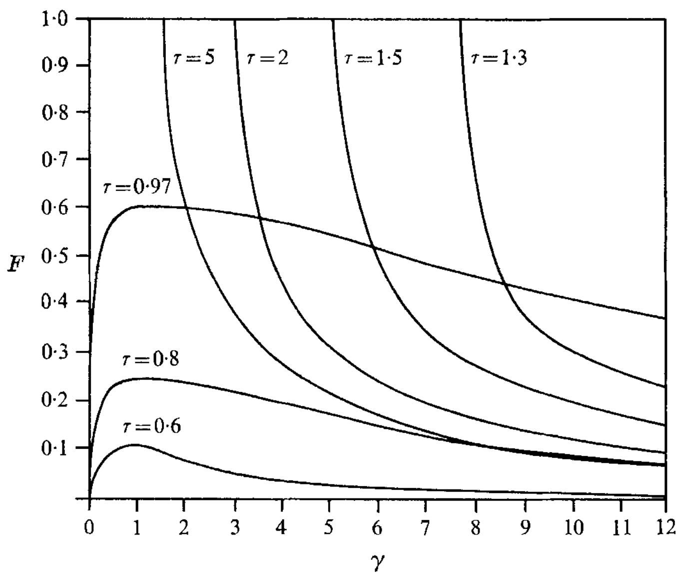

FIGURE 2. The flux ratio F of diffracted energy to incident energy as it varies with the discontinuity depth ratio γ for various values of τ, the wave period in pendulum days.

which would now be more easily diverted from following the coastline are those with a long wave period, their energy being propagated along any abrupt changes in depth present near the coastline. If these depth changes then diminish and disappear, then as Longuet-Higgins infers, the energy would eventually appear in the record of the horizontal currents alone, instead of being shared with the surface elevations in the form of potential energy.

As $\epsilon \to 0$ and $\tau \to (\gamma+1)/(\gamma-1)$ the conditions for the existence of the double Kelvin waves becomes more critical, as does the sharing of energy flux between the two types of wave. The double Kelvin response is becoming more difficult to incite, and consequently more energy is being diffracted. As this critical region is approached the long wave theory is becoming less valid. The double Kelvin waves predicted have progressively smaller wavelengths, i.e. $2\pi/m = 2\pi(c_1/f)\epsilon^{\frac{1}{2}}$, and the wave disturbance is being concentrated progressively nearer the discontinuity with larger amplitudes at $y = 0$ as may be inferred from (6.10). It can also be shown that the energy associated with the double Kelvin wave in the critical region is large and $O(1/\epsilon)$, consisting mostly of kinetic energy rather than potential energy. However, there is no violation of the energy condition (7.4) since it can be shown that this large energy travels along the depth discontinuity with a correspondingly smaller group velocity c_G of $O(\epsilon^{\frac{3}{2}})$.

To conclude, the above results may now be grouped together. The general pattern which emerges from the details of the particular model chosen here is that those Kelvin waves which are the most persistent in diffracting along irregularly shaped coastlines bounding a sea of abruptly varying depth are of intermediate wave period, in the range $0{\cdot}9 < \tau < 3$ approximately.

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